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# INTERACTION OF SPHERICAL SOURCE FLOW AND AXISYMMETRIC FREE JET WITH A RAREFIED BACKGROUND

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by

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13. ABSTRACT

The interaction of a hypersonic source flow and an axisymmetric free jet with a rarefied background is investigated. Based on the singular nature of the distribution function, the Boltzmann equation is split into two kinetic equations. This approach includes the basis for the "beam-continuum" method as a special case. Application of the simple moment method leads to a closed form solution for the results arrived at by Muntz, Hamel and Maguire for source flow. The solution for the axisymmetric free jet also agrees with the experimental results due to Brook, Hamel and Muntz.

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# I. INTRODUCTION

The effect of rarefied background gas on the hypersonic source flow has been investigated by Muntz, Hamel and Maguire[1] (MHM) and Brook and Hamel[2] (BH). The earlier authors (MHM) made certain ad hoc assumptions regarding the flow processes and were able to give a simple picture of the plume rarefaction. This theory was modified by the later authors (BH), who used a kinetic model as represented by the Boltzmann equation and a three-fluid model comprised of jet, scattered, and background molecules. In their analysis the Boltzmann equation was arbitrarily split into three equations describing the distribution function for each type of molecule, and formal asymptotic solutions were sought in terms of a rarefaction parameter.

In the present report, we utilize the fact that the distribution function of the jet molecules is singular in velocity space. Subtracting this explicit singularity leads to the splitting of the Boltzmann equation into two kinetic equations whose asymptotic solution for a rarefied background is sought. A similar procedure by Grad[3] has been successfully applied to obtain the structure of a strong shock wave. As one alternative for carrying out a solution to these equations, an approximate technique of moments, following the well-known method of Mott-Smith, may be employed. It is shown that this approach produces a conveniently simple closed-form solution of reasonable accuracy. Indeed, it is shown that the previously-cited ad hoc theory of MHM is actually equivalent to that due to the moment-method solution of the present equations.

The theory is applied here to two sample cases: that of a spherical source interacting with a rarefied background and that of a free axisymmetric jet interacting with a rarefied background.

It is of further interest to point out that the present analysis produces as a special case the beam-continuum equations derived by Rott and

Whittenbury[4], which have been applied by them to an analysis of the structure of a strong shock wave and to the evaluation of sphere drag in hypersonic flow. These equations have recently been further applied by Kot and Turcotte[5] to the solution of the flow over the leading-edge of a sharp plate. The particular form of the beam-continuum equations derivable from the present analysis depends upon the choice of the continuum part of the distribution function. Some of the alternative forms for the distribution function are that due to Mott-Smith or Chapman-Enskog. The Mott-Smith form produces a special form of the continuum model which permits a closed form solution to the problem. The Chapman-Enskog form leads to the Navier-Stokes model for the continuum portion and requires a numerical solution of it. It may be noted that a systematic expansion based on the Chapman-Enskog assumption will produce the present Mott-Smith-based solution as the leading approximation.

Following Brook and Hamel, we assume here that the flow has already expanded such that the velocity has achieved its limiting value before an appreciable effect of the background is experienced.

## II. BASIC EQUATIONS

The Boltzmann equation for hard sphere molecular interactions can be written as

$$\frac{Df}{Dt} = \iiint (f'f_1' - ff_1)\, g b\, db\, d\epsilon\, d\vec{\xi}_1 = J(f,f)$$

$$= G(f,f) - \nu(\vec{r},\vec{\xi})f \qquad (1)$$

where f is the distribution function, $\vec{\xi}$ is the molecular velocity vector, g is the relative speed of the colliding molecules, $\nu$ is the collision frequency, b and $\epsilon$ are two geometric variables, $f_1$ is the value of f with $\vec{\xi}$ replaced by $\vec{\xi}_1$, the velocity vector of the second particle. $f'$ and $f_1'$ are the values of f for velocities of two molecules after collisions. We

write

$$f = f_j + F \tag{2}$$

where $f_j$ is the distribution function for the jet molecules. Since the jet has already expanded to its limiting value, we may write $f_j$ as

$$f_j = n_j(\vec{r})\,\delta(\xi - u_o)\,\delta(\eta)\,\delta(\zeta) \ ,$$

$\delta$ being the Dirac delta function. F corresponds to the continuum part of the distribution and approaches the background value $f_b$ at large distances from the source. Substituting this into Eq. (1) leads to

$$\frac{Df_j}{Dt} = -\nu_F f_j + J(f_j, f_j)$$

and (3)

$$\frac{DF}{Dt} = -\nu_j F + 2J(f_j, F) + J(F,F)$$

where

$$J(f_j, F) = \int f_j' F_1' g b\,db\,d\epsilon\,d\vec{\zeta}$$

$$\nu_F = \int F_1 g b\,db\,d\epsilon\,d\vec{\zeta}$$

$$\nu_j = \int f_j g b\,db\,d\epsilon\,d\vec{\zeta} \quad \text{and} \quad \nu = \nu_j + \nu_F$$

We nondimensionalize the various quantities as

$$\frac{u_o}{\sqrt{2RT_o}} = V_\infty \ ; \quad \frac{r}{r^*} = R \ ; \quad g/V_{r_\infty} = \bar{g}$$

$$(\vec{\zeta}, \vec{\xi})/(2RT_b)^{1/2} = \vec{\zeta}, \vec{\xi} \ ; \quad F = \frac{n_{b_\infty}}{(2RT_b)^{3/2}} \bar{F}$$

$$\frac{n_j}{n_j^*} = \vec{n}_j \ ; \quad \frac{b}{\sigma} = \bar{b} \ ; \quad \frac{\epsilon}{\pi} = \bar{\epsilon} \ .$$

Thus the Eqs. (3) become

$$\frac{D\bar{f}_j}{Dt} = \frac{V_{r_\infty} \pi\sigma^2 r^* n^*}{(2RT_b)^{\frac{1}{2}}} \left[\frac{n_{b_\infty}}{n^*} (-\bar{\nu}_F \bar{f}_j) + J(\bar{f}_j, \bar{f}_j)\right]$$

$$\frac{D\bar{F}}{Dt} = \frac{\pi V_{r_\infty} \sigma^2 n^* r^*}{(2RT_b)^{\frac{1}{2}}} \left[-\bar{\nu}_j \bar{F} + \frac{n_{b_\infty}}{n^*} J(F,F) + 2J(\bar{f}_j, \bar{F})\right] .$$

Let

$$R_p = \frac{\pi V_{r_\infty} \sigma^2 n^* r^*}{(2RT_b)^{\frac{1}{2}}} \tag{4}$$

where $n^*$ is the sonic number density, $r^*$ the sonic radius, $n_{b_\infty}$ and $T_b$ are the number density and temperature of the background gas and $V_{r_\infty}$ is a representative relative collision velocity. In these equations, $n_{b_\infty}/n^*$ and $R_p$ are the two parameters, the latter being proportional to inverse Knudsen number. Since we have assumed that the flow has already expanded to its final hypersonic value, we rescale the variables as

$$R_p^2 \bar{f}_j = f_j^* \quad ; \quad \bar{n}_j R_p^2 = N_j$$

$$R/R_p = \tilde{R}$$

Now, taking the limit $n_{b_\infty}/n^* \to 0$ and $R_p \to \infty$, we get to the lowest order,

$$\frac{Df_j^*}{Dt} = \frac{1}{R_p^2} J = 0 \qquad \text{(i)}$$

$$\frac{D\bar{F}}{Dt} = -\bar{\nu}_j \bar{F} + 2J(f_j^*, F) \qquad \text{(ii)} \tag{5}$$

Eq. (5-ii) provides a clear picture of the limit to perfect vacuum expansion. $\bar{F} \equiv 0$ can be one of the solutions only if the background distribution function is identically zero. Even a small value for $f_b$ leads to a finite value of $\bar{F}$.

In the next section, a moment method is presented to solve Eq. (5).

## III. MOMENT METHOD

By assuming $\bar{F}$ to be a shifted Maxwellian, Grad[3] was able to predict the structure of a strong shock wave. In the present case, we assume the background distribution function $\bar{F}$ to contain a variable number density $N_b$ and to be of the nondimensional form,

$$\bar{F} = \frac{N_b}{\pi^{3/2}} e^{-c^2} \quad . \tag{6}$$

Substituting this in Eq. (5-ii) and multiplying by $\xi$ and integrating over the velocity space, we get

$$\frac{1}{\tilde{R}^2} \frac{d}{d\tilde{R}}(\tilde{R}^2 N_b) - \frac{2N_b}{\tilde{R}} = 2\iiint \xi J(f_j^*, F)\, d\bar{\xi} \tag{7}$$

The right-hand side of Eq. (7) is easily evaluated by realizing the fact that collisions must conserve momentum for the gas represented by $(f_j+F)$. So from Eq. (4), we get

$$2\langle \xi J(f_j^*, \bar{F})\rangle = \langle \xi \bar{\nu}_F f_j^* \rangle \tag{7a}$$

[$\bar{\nu}_F f_j^*$ term is small as compared to terms in Eqs. (3-i) or (4-i) but is of the same order as other terms in Eqs. (3-ii) and (4-ii).] Thus, Eqs. (5), (7) and (7a) lead to

$$V_\infty N_j \tilde{R}^2 = \text{const.} = (5/8\ \tau)^{1/2} \tag{8}$$

$\tau = T_{stag}/T_b$ , $T_{stag}$ being the stagnation temperature

and

$$\frac{1}{\tilde{R}^2} \frac{d}{d\tilde{R}} (\tilde{R}^2 N_b) - \frac{2N_b}{\tilde{R}} = \langle \xi \bar{\nu}_F f_j^* \rangle$$

or

$$\frac{dN_b}{d\tilde{R}} = N_j V_\infty \nu_F^* N_b \tag{9}$$

where

$$\nu_F^* = \frac{(2RT_b)^{\frac{1}{2}}}{V_{R_\infty}}\left[\frac{1}{\sqrt{\pi}}e^{-V_\infty^2} + \frac{1+2V_\infty^2}{2V_\infty}\operatorname{erf}(V_\infty)\right] \tag{10}$$

Eqs. (8) and (9) lead to

$$\frac{1}{N_b}\frac{dN_b}{d\bar{R}} = \frac{1}{\bar{R}^2}\nu_F^*$$

the solution of which, satisfying the boundary conditions at infinity, comes out to be

$$N_b = \exp(-\nu_F^*/\bar{R}) \tag{11}$$

Eq. (9) and the solution (11) are the same as obtained by MHM. Use of a shifted Chapman-Enskog distribution function for $\bar{F}$ leads to the well-known beam-continuum equations of Rott and Whittenbury which could provide a better approximation to the value of $N_b$.

In order to find the structure of the distribution function, one has to integrate Eq. (5-ii). In its present form it is very difficult to solve Eq. (5-ii) without resort to some sort of approximations. Since the collisional invariant moments of $J(f_j,F)$ have to satisfy relation (7a), we will replace $J(\bar{f}_j,\bar{F})$ in (5-ii) by $\bar{\nu}_F\bar{f}_j$ ($\nu_F$ is further assumed to be a known function of velocity). Actually a better approximation could be to evaluate this term by using the solution of the moment method given in the last section. Also, in order to avoid unpleasant singularities in $J(\bar{f}_j,\bar{F})$, we will replace the Ansatz by the following:

$$f = \frac{n_j(r)}{(2\pi RT_{11})^{\frac{1}{2}}}e^{-(\xi-u_0)^2/2RT_{11}}\ \delta(\cdot)\delta(\cdot)+F$$

where $T_{11}$ is the constant value of the temperature at which the freeze takes place. The splitting of the Boltzmann equation still holds and

variation of $\bar{F}$ is still governed by Eq. (5-ii). Now, $\bar{\nu}_j$ can be written as

$$\bar{\nu}_j = N_j(\tilde{R}) \frac{(2RT_b)^{1/2}}{V_{r_\infty}} \left[\frac{1}{\sqrt{\pi}} e^{-\mathcal{C}_1^2} + \frac{1+2\mathcal{C}_1^2}{2\mathcal{C}_1} \operatorname{erf}(\mathcal{C}_1)\right]$$

$$= N_j \nu^*_j \tag{12}$$

where

$$\mathcal{C}_1 = \left(\frac{2RT_b}{2RT_{11}}\right)^{1/2} (\xi - V_\infty) \quad ; \quad c^2 = \mathcal{C}_1^2 + \mathcal{C}_2^2 + \mathcal{C}_3^2 \ .$$

Following Brook and Hamel, Eq. (5) can be integrated along the characteristic line and we get

$$\bar{F} = f_b e^{-\frac{\nu_j^*}{2\tilde{R}c}\frac{\theta}{\sin\theta}} + \frac{\nu_F^* f_j^*}{2\tilde{R}^2 c\sin^2\theta} \int_{\tilde{R}\cos\theta}^{\infty} \frac{N_b e^{-\frac{\nu_j^*}{2\tilde{R}c\sin\theta}\left[\tan^{-1}\frac{\rho'}{\tilde{R}\sin\theta} - \left(\frac{\pi}{2}-\theta\right)\right]}}{\left[1+(\rho'/\tilde{R}\sin\theta)^2\right]} d\rho'$$

The last equation can be put in a simple form if we write

$$\rho' = \tilde{R}\sin\theta\tan Z.$$

Thus,

$$\bar{F} = f_b \exp\left(-\frac{\nu_j^*}{2\tilde{R}c}\frac{\theta}{\sin\theta}\right) + \frac{\nu_F^* f^*_j}{2\tilde{R}c\sin\theta} e^{\frac{\nu_j^*}{2\tilde{R}c}\frac{\frac{\pi}{2}-\theta}{\sin\theta}} \int_{\frac{\pi}{2}-\theta}^{\pi/2} N_b e^{-\nu_j^* Z/2\tilde{R}c\sin\theta} dZ$$

Integration over velocity space leads to a linear integral equation for $N_b$. An iterative solution is immediately suggested by the nature of the equation. However, simple estimates can be obtained by assuming $N_b$ to be independent of Z. For certain types of collisions this can be a fairly good approximation and has been used by Kogan[8] and others. Thus, we get

$$\bar{F} = f_b \exp\left(-\frac{\nu_j^*}{2\tilde{R}c}\frac{\theta}{\sin\theta}\right) - \frac{\nu_F^*}{\nu_j^*} f_j N_b \left[\exp\left(-\frac{\nu_j^*}{2\tilde{R}c}\frac{\theta}{\sin\theta}\right) - 1\right] \tag{13}$$

If the last term in Eq. (13) is interpreted as due to scattered molecules, we can easily see that their contribution is positive and of the order of $(\nu_F^*/\nu_j^*)$. Apart from velocity dependent collision frequency, the present result differs from that of Brook and Hamel in the expression for scattered molecules. Integration of Eq. (13) over velocity space leads to the value of $N_b$.

## IV. AXISYMMETRIC JET

Once again we will assume that the jet has already expanded to its final hypersonic value before the effect of background molecules becomes important. So Eq. (5) still applies to the present problem. With a spherical coordinate system $(r, \theta)$, the solution of Eq. (5-i) is

$$f_j = n_j(r,\theta)\,\delta(\xi-u_o)\,\delta(\eta)\,\delta(\zeta) \quad . \tag{14}$$

The results of studies on axisymmetric jets by Robertson and Willis[6], and Grundy[7], indicate that the number density is essentially given by the inviscid solution and the temperature freezes at different values along different streamlines. For number density $n_j$ one can either use the density evaluated from the similarity solution by Thornhill[9] or the co-relation with the characteristics solution, due to Ashkenas and Sherman[10], can be used. In the present case the latter value for $n_j$ is used. Thus,

$$n_j(r,\theta) = \frac{\text{const.}}{r^2}\cos^2\frac{\pi\theta}{2\varphi} \quad ; \quad \varphi = 1.365 \tag{15}$$

A straightforward application of Mott-Smith moment technique to Eq. (5-ii) leads to

$$\frac{1}{N_b}\,\frac{2N.}{2\tilde{R}} = \frac{\nu_F^*}{\tilde{R}^2}\cos^2\frac{\pi\theta}{2\varphi}$$

The integration leads to

$$N_b = A_1(\theta)\exp\left(-\frac{\overset{*}{F}}{\tilde{R}}\cos^2\frac{\pi\theta}{2\phi}\right)$$

In order to satisfy the boundary condition at $\tilde{R}\to\infty$, we must have $A_1 = 1$. Thus,

$$N_b = \exp - \frac{\overset{*}{F}}{\tilde{R}}\cos^2\frac{\pi\theta}{2\phi} \qquad (16)$$

On the jet centerline where $\theta=0$, (16) reduces to Eq. (11). So far the jet flow is hypersonic; the '$\theta$' variation of freeze temperature does not influence Eq. (16) to a first order. Also, this simple expression for variation of $N_b$ suggests that background molecules are moving radially and the penetration distance into the jet is different along different streamlines. The latter result is intuitively reasonable as the attenuation of the background depends upon the availability of the number of jet molecules. As a consequence, the background penetrates farther along streamlines away from the axis of the jet. This solution agrees with the experimental data obtained by Brook, Hamel and Muntz.[+]

## V. CONCLUSION

The interaction of a hypersonic source flow and an axisymmetric free jet with a rarefied background has been investigated. Based on the singular nature of the distribution function, the Boltzmann equation is split into two kinetic equations. This approach includes the basis for the "beam-continuum" method as a special case. Application of the simple moment method leads to a closed form solution for the results arrived

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[+]An extended abstract of a paper presented by Brook, Hamel and Muntz at the 8th Rarefied Gas Dynamics Symposium, and kindly made available by Dr. Brook, mentions, "If the background density is assumed to decay exponentially , $n_b = n_{b\infty}\exp(-R_p(\theta)/R)$, and if we calculate $R_p(\theta)$, we can see that the theoretical results, and data both agree with the approximation: $R_p(\theta) = R_p(0)\cos^2 1.15\theta$.

at by MHM for source flow. The solution for the axisymmetric free jet also agrees with the experimental results due to Brook, Hamel and Muntz.

## VI. REFERENCES


1. Muntz, E.P., Hamek, B.B. and Maguire, B.L.: Some Characteristics of Exhaust Plume Rarefaction. AIAA J., 8, pp. 1651-1658, 1970.
2. Brook, J.W. and Hamel, B.B.: Spherical Source Flow with a Finite Back Pressure. Grumman Aerospace Corp., Research Report RE-427J (to appear in The Physics of Fluids).
3. Grad, H.: Singular and Non-uniform Limits of Solutions of the Boltzmann Equation. Proceedings of the Symposium (on Transport Theory) in Applied Mathematics, 1, pp. 269-308, 1967.
4. Rott, N. and Whittenbury, C.G.: A Flow Field Model for Hypersonic Rarefied Gas Dynamics with Applications to Shock Structure and Sphere Drag. Douglas Aircraft Co., Report No. SM-38524, 1961.
5. Kot, S.C. and Turcotte, D.L.: Beam Continuum Model for Hypersonic Flow Over a Flat Plate. AIAA J., 10, pp. 291-295, 1972.
6. Robertson, S.J. and Willis, D.R.: Method-of-Characteristics Solution of Rarefied, Monoatomic Gaseous Jet Expansion into a Vacuum. AIAA J., 9, pp. 291-296, 1971.
7. Grundy, R.E.: Axially Symmetric Expansion of a Monoatomic Gas From an Orifice into a Vacuum. Phys. Fluids, 12, pp. 2011-2018, 1969.
8. Kogan, M.N.: Rarefied Gas Dynamics. Plenum Press, New York, 1969.
9. Thornhill, C.K., Royal Armament Research and Development Report No. 6/65, 1965.
10. Askenas, H. and Sherman, F.S.: The Structure and Utilization of Supersonic Free Jets in Low Density Wind Tunnels. "Rarefied Gas Dynamics", J.H. deLeeuw, ed., II, pp. 84-105, 1966.